Caringbah High School

Year 12 2022
Mathematics Extension 1
HSC Course
Assessment Task 4

General Instructions
- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- NESA-approved calculators may be used
- A reference sheet is provided
- In Questions 11-14, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for partial or incomplete answers

Total marks – 70

Section I **10 marks**
Attempt Questions 1-10
Mark your answers on the answer sheet provided. You may detach the sheet and write your name on it.

Section II **60 marks**
Attempt Questions 11-14
Write your answers in the answer booklets provided. Ensure your name or student number is clearly visible.

Name: ____________________________ **Class:** ____________

Marker's Use Only						
Section I	**Section II**				**Total**	
Q 1-10	**Q11**	**Q12**	**Q13**	**Q14**		
/10	/15	/15	/15	/15	/70	%

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10

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1. $\cos^2\left(\frac{\pi}{4}-x\right)-\sin^2\left(\frac{\pi}{4}-x\right)$ simplifies to

(A) $\cos 2x$

(B) $\sin 2x$

(C) $\cos x$

(D) $\sin x$

2. Which of the following integrals is obtained when the substitution $u=(\log_e x)^2$ is applied to

$$\int_e^{e^2} \frac{(\log_e x)^3}{x}\,dx \ ?$$

(A) $\frac{1}{2}\int_1^4 u\,du$

(B) $2\int_1^4 u\,du$

(C) $\frac{1}{2}\int_{e^2}^{e^4} u\,du$

(D) $\int_1^4 u^6\,du$

3. What is the acute angle between the vectors $\underset{\sim}{i} + 2\underset{\sim}{j}$ and $4\underset{\sim}{i} + 2\underset{\sim}{j}$, correct to the nearest degree?

(A) 18°

(B) 26°

(C) 32°

(D) 37°

4. Which of the following represents the inverse of $f(x) = e^{2x} + 2$?

(A) $f^{-1}(x) = \dfrac{1}{e^{2x} + 2}$

(B) $f^{-1}(x) = \dfrac{\log_e(x-2)}{2}$

(C) $f^{-1}(x) = \dfrac{\log_e x - 2}{2}$

(D) $f^{-1}(x) = \log_e(x-2)$

5. What is the value of the term independent of x in the expansion of $\left(x^2 + \dfrac{2}{x}\right)^6$?

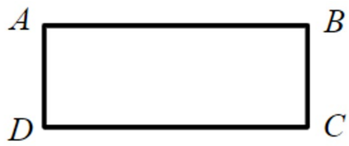
(A) 60

(B) 160

(C) 192

(D) 240

6. Consider the rectangle below.



Which of the following statements is FALSE?

(A) $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

(B) $\overrightarrow{AB}-\overrightarrow{BC}=\overrightarrow{DB}$

(C) $\overrightarrow{AB}+\overrightarrow{DC}=\overrightarrow{AC}$

(D) $\overrightarrow{AB}\cdot\overrightarrow{BC}=\overrightarrow{AC}$

7. A projectile has the equation of path $y=-3x^2+2x+4$.
How far will it have travelled horizontally before it returns to its initial height?

(A) $\frac{1}{3}$ units

(B) $\frac{2}{3}$ units

(C) 2 units

(D) 4 units

8. 5 boys and 5 girls are seated at a round table. In how many ways can this happen if a boy named Robert and a girl named Hannah are sitting together?

(A) 725760

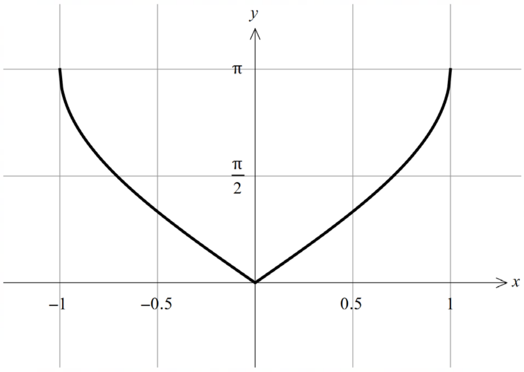
(B) 362880

(C) 40320

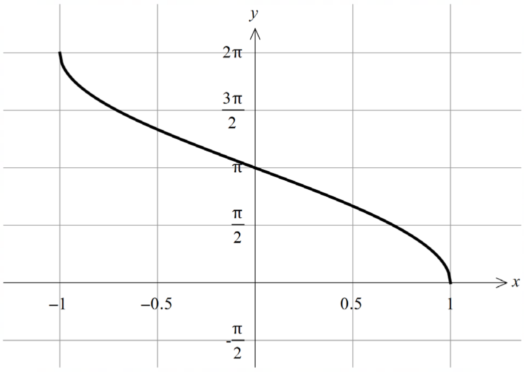
(D) 80640

9. Which graph best represents $y = \left|2\cos^{-1} x - \pi\right|$?

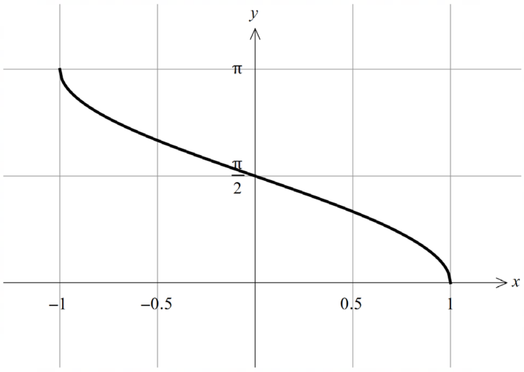
(A)



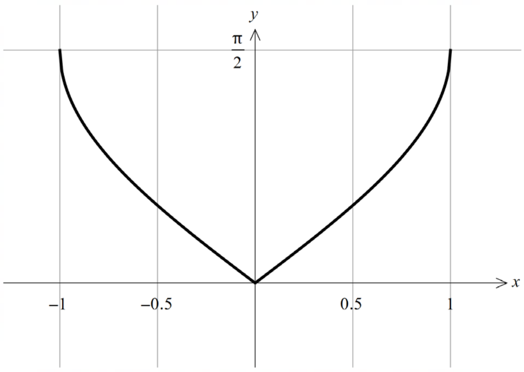
(B)



(C)



(D)



10. Given that α, β, γ are roots of $3x^3 - 2x^2 + x - 1 = 0$,

then $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ is equal to:

(A) -1

(B) 2

(C) 1

(D) -2

End of Section I

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour 45 minutes for this section

Answer each question in a SEPARATE writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Solve the inequality **2**

$$\frac{1}{x-1} < 1$$

(b) The polynomial $P(x) = x^3 - 4x^2 + tx + 10$ has roots $\alpha, -\alpha$ and β. **3**
Find the three roots and hence find the value of *t*.

(c) Use the substitution $u = x + 1$ to evaluate **3**

$$\int_1^3 \frac{x+2}{(x+1)^2}\,dx$$

Express your answer in simplest exact form.

(d) Given $\underset{\sim}{u} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{v} = -2\underset{\sim}{i} + 4\underset{\sim}{j}$ **2**
Find $proj_{\underset{\sim}{u}}\underset{\sim}{v}$

Question 11 continues on page 8

(e) Records show that 20% of students at a school play a musical instrument. A sample of 100 students at the school is to be taken to determine the proportion who play a musical instrument.

(i) Find the mean and standard deviation of the distribution of such a sample proportion. **2**

(ii) Use the following extract of the table of values of $P(Z \leq z)$, where Z has a standard normal distribution, to estimate the probability that a sample of 100 students at the school contains at least 18 and at most 26 who play a musical instrument. **3**

STANDARD NORMAL DISTRIBUTION TABLE

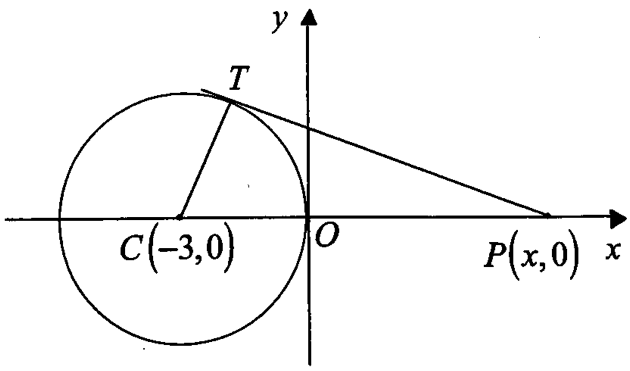
Entries represent $P(Z \leq z)$. The value of z to the first decimal is given in the left column. The second decimal is given in the top row.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) $P(x, 0)$ is a point on the positive x-axis. T is the point of contact of a tangent drawn from P to the circle with centre $C(-3, 0)$ and radius 3.



(i) Show that the length of $PT = \sqrt{x^2 + 6x}$ **1**

(ii) If the units in the above diagram are cm, and P is moving along the x-axis away from O at a constant rate of $0.1\,\text{cm s}^{-1}$, find the rate of change of PT when $x = 2$ cm. **2**

(b) Consider the function $f(x) = 2\cos^{-1}(x-1)$ where $1 \le x \le 2$

(i) Sketch the curve $y = f(x)$, showing clearly the endpoints. **2**

(ii) Find the equation of the inverse function $f^{-1}(x)$ and state its domain. **2**

(c) Given $\underset{\sim}{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

(i) Find $\underset{\sim}{a} + \underset{\sim}{b}$ and $\underset{\sim}{a} - \underset{\sim}{b}$ **2**

(ii) Hence find $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b})$ **1**

(iii) State if $\underset{\sim}{a} + \underset{\sim}{b}$ is perpendicular to $\underset{\sim}{a} - \underset{\sim}{b}$ and justify your answer. **1**

Question 12 continues on page 10

(d) **(i)** Use the substitution $x = \tan^2\theta$ to show that **2**

$$\int_0^1 \frac{\sqrt{x}}{(1+x)^2}\,dx = \int_0^{\frac{\pi}{4}} 2\sin^2\theta\,d\theta$$

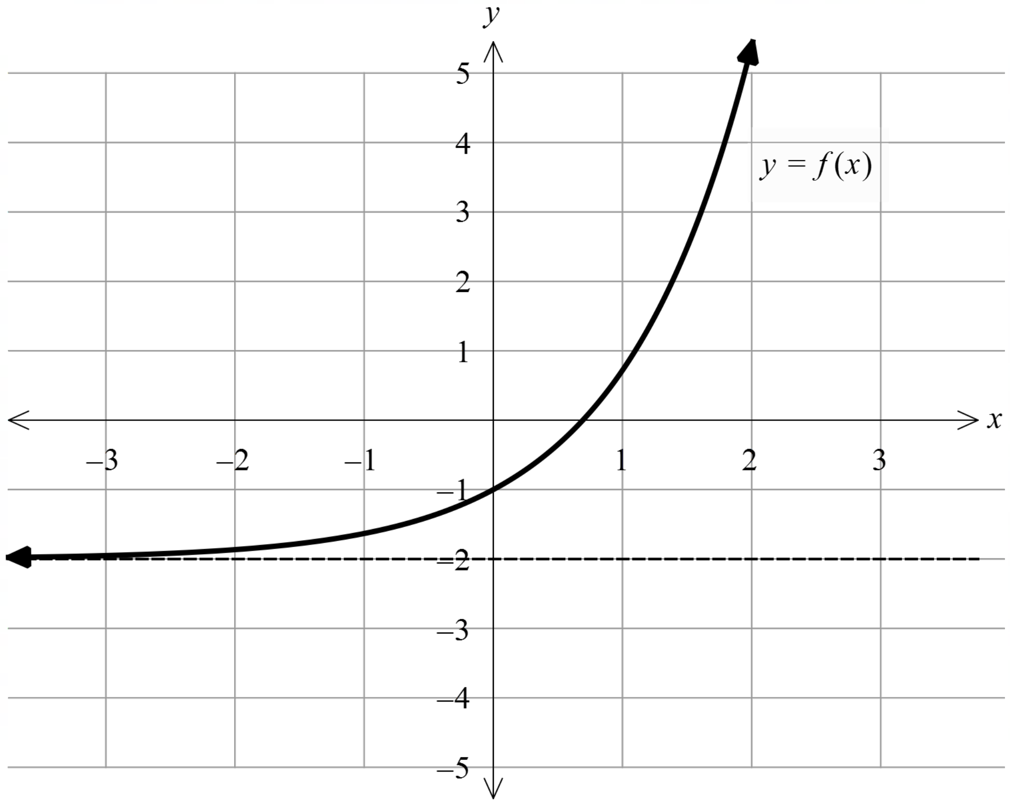
(ii) Hence, or otherwise, find in simplest exact form the value of **2**

$$\int_0^1 \frac{\sqrt{x}}{(1+x)^2}\,dx$$

End of Question 12

Question 13 (5 marks) Use a SEPARATE writing booklet.

(a) The diagram below shows the graph of $f(x) = e^x - 2$ and it's horizontal asymptote $y = -2$



On separate diagrams, on the templates provided, sketch the following graphs, in each case showing the intercepts on the axes and the equations of any asymptotes.

(i) $y = \left(f(x)\right)^2$ **1**

(ii) $y = \log_e f(x)$ **1**

(iii) $y = \dfrac{1}{f(x)}$ **2**

(iv) $y^2 = \left|f(x)\right|$ **2**

Question 13 continues on page 12

(b) **(i)** Show that $2\cos\left(x+\frac{\pi}{3}\right)=\cos x-\sqrt{3}\sin x$ **1**

(ii) Hence, or otherwise, solve the equation **2**

$$\cos x-\sqrt{3}\sin x=\sqrt{3} \text{ for } 0\le x\le 2\pi$$

(c) Newton's Law of Cooling states that when an object at temperature T°C is placed in an environment at temperature T_0°,
the rate of temperature loss is given by the equation

$$\frac{dT}{dt}=k(T-T_0)$$

Where *t* is the time in seconds and *k* is a constant.
A packet of peas, initially at 24°C, is placed in a snap-freeze refrigerator in which the internal temperature is maintained at -40°C.

(i) Show that $T=-40+64e^{kt}$, representing the above data, is a solution to the equation . **1**

(ii) After 5 seconds, the temperature of the packet is 19°C. **2**
How long will it take for the packets temperature to reduce to 0°C?

(d) Given that $f(x)=\cos^{-1}(2x-1)-2\cos^{-1}\sqrt{x}$, show that $f'(x)=0$ **3**

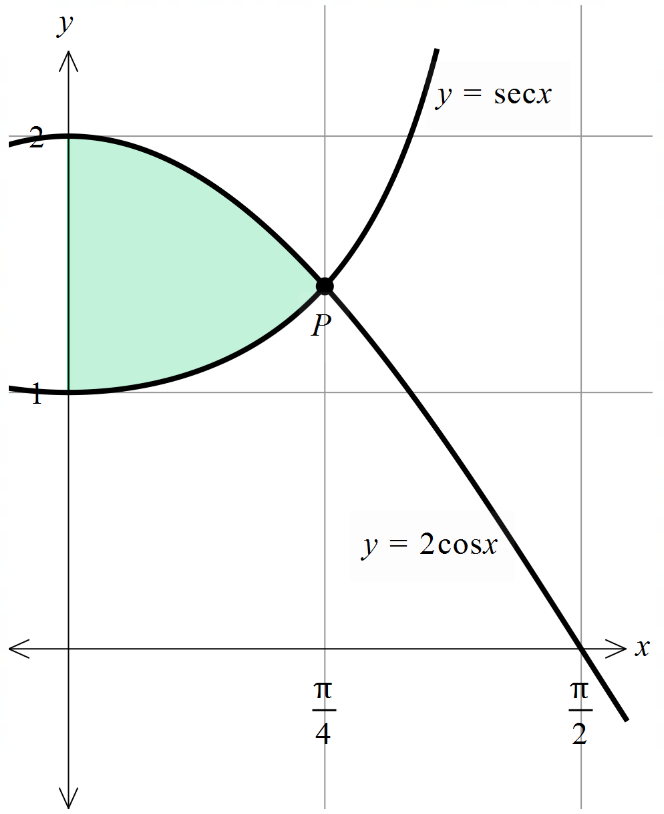
End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Use Mathematical Induction to show that for all positive integers $n \geq 2$, **3**

$$2 \times 1 + 3 \times 2 + 4 \times 3 + ... + n(n-1) = \frac{1}{3}n(n^2-1)$$

(b) P is the point of intersection between $x = 0$ and $x = \frac{\pi}{2}$ of the graphs of $y = \sec x$ and $y = 2\cos x$, as shown.

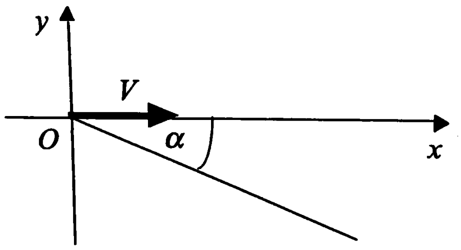


(i) Verify that the x-coordinate of P is $\frac{\pi}{4}$ **1**

(ii) The shaded region makes a revolution about the x-axis. Show that the volume of the resulting solid is $\frac{\pi^2}{2}$ cubic units. **3**

Question 14 continues on page 14

(c)



A particle is projected horizontally from a point O with speed V ms^{-1} down a slope which is inclined at an angle $\alpha = \tan^{-1}\frac{1}{2}$ below the horizontal. The particle moves in a vertical plane under gravity where the acceleration due to gravity is $g\,\text{ms}^{-2}$.
At time t seconds the horizontal and vertical displacements from O, x metres and y metres respectively, are given by

$$x = Vt$$
$$y = -\frac{1}{2}gt^2.$$

(DO NOT PROVE THESE RESULTS)

(i) Show that the particle hits the slope after time $\frac{V}{g}$ seconds. **2**

(ii) Show that the particle hits the slope with velocity $V\sqrt{2}$ ms^{-1} at an angle of 45° to the vertical. **2**

(d) **(i)** Write down the Binomial expansion of $(1-x)^{2n}$ in ascending powers of x **1**

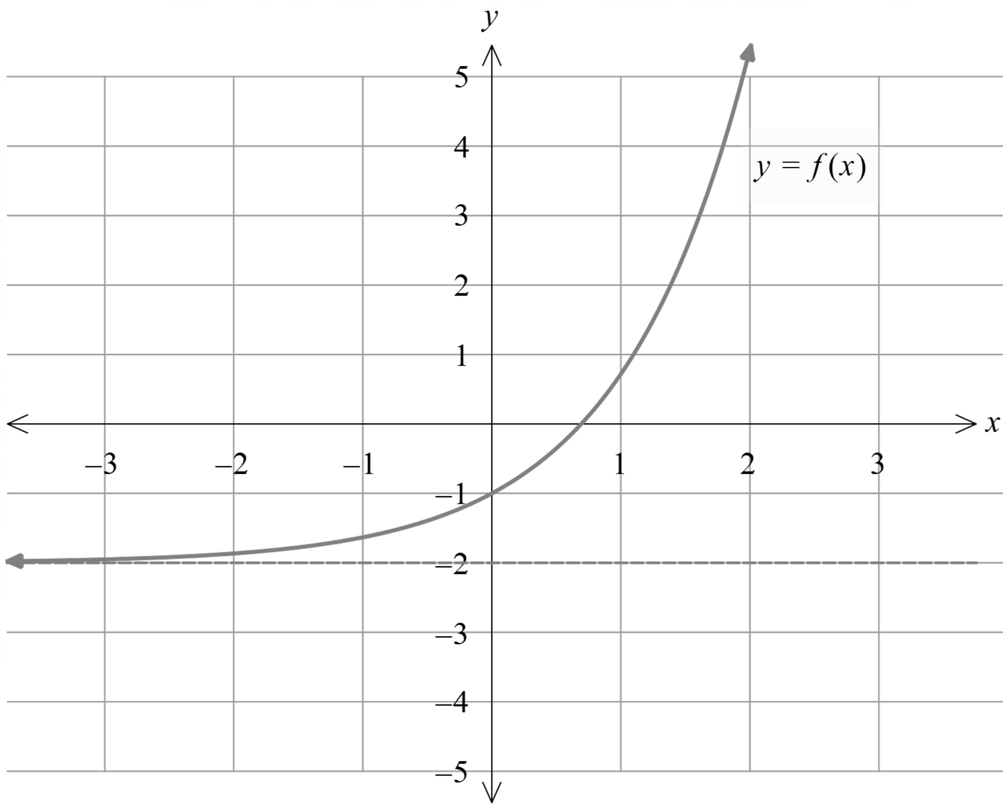
(ii) Hence show that **3**

$$\binom{2n}{1} + 3\binom{2n}{3} + \ldots\ldots\ldots + (2n-1)\binom{2n}{2n-1} = 2\binom{2n}{2} + 4\binom{2n}{4} + \ldots\ldots + 2n\binom{2n}{2n}$$

End of Paper

Templates for Q13 Answer Booklet

Q13(a)(i) $y = \left(f(x)\right)^2$



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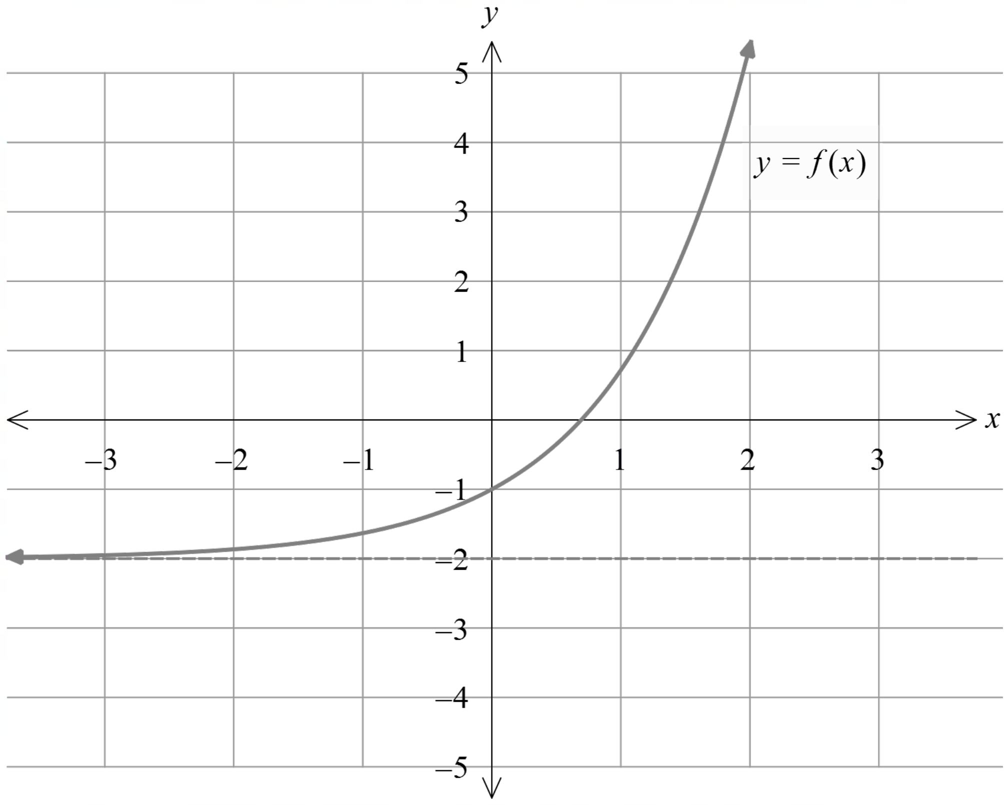
Q13(a)(ii) $y = \log_e f(x)$



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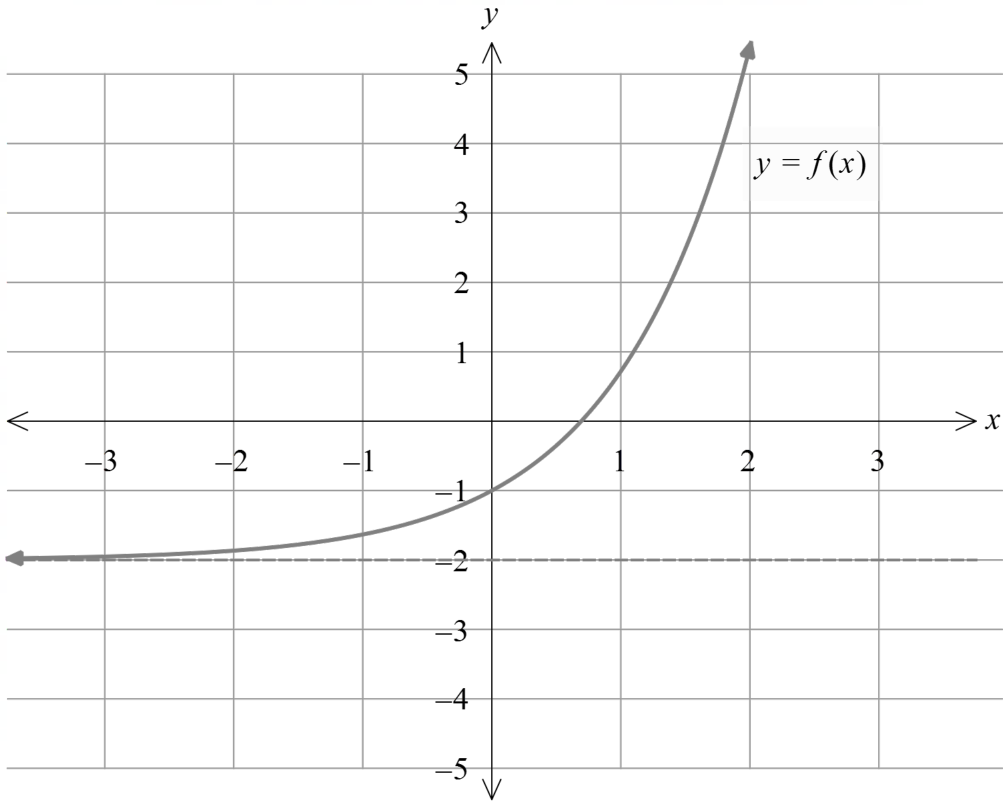
Q13(a)(iii) $y = \dfrac{1}{f(x)}$



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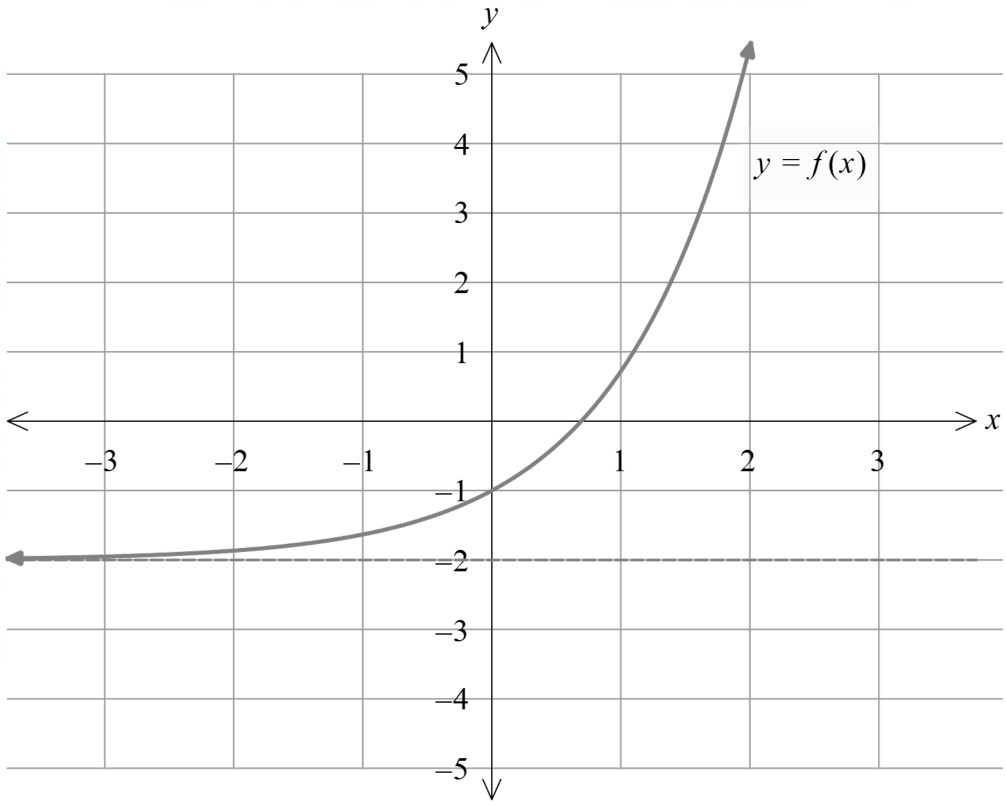
Q13(a)(iv) $y^2 = |f(x)|$



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Mathematics Extension 1 - Trial HSC 2022 Solutions

Section 1

1. B	2. A	3. D	4. B	5. D
6. C/D	7. B	8. D	9. A	10. C

Section 2

Question 11

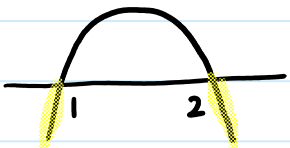
(a) $\frac{1}{x-1} < 1$

$\frac{1}{x-1} - 1 < 0$

$\frac{1-(x-1)}{x-1} < 0$

$\frac{2-x}{x-1} < 0$ $\quad \times (x-1)^2$ both sides

$(x-1)(2-x) < 0$



$\therefore x < 1, \ x > 2$

(b) $P(x) = x^3 - 4x^2 + tx + 10$, roots $\alpha, -\alpha, \beta$

$\sum \alpha = -b/a$

$\alpha + -\alpha + \beta = -4/1$

$\beta = -4$

$\sum \alpha\beta\gamma = -d/a$

$-4\alpha^2 = -10/1$

$\alpha^2 = 5/2$

$\alpha = \pm\sqrt{5/2} = \pm\sqrt{10}/2$

$\sum \alpha\beta = c/a$

$-\alpha^2 + \alpha\beta - \alpha\beta = t/1$

$t = -5/2$

$\therefore$ Roots are $-4, \pm\sqrt{10}/2$ and $t = -5/2$

(c) $\displaystyle\int_1^3 \frac{x+2}{(x+1)^2}\,dx$

Let $u = x+1$

$du = dx$

$\begin{cases} x=1, u=2 \\ x=3, u=4 \end{cases}$

$= \displaystyle\int_2^4 \frac{u+1}{u^2}\,du$

$= \displaystyle\int_2^4 u^{-1} + u^{-2}\,du$

$= \left[\ln u + \frac{u^{-1}}{-1}\right]_2^4$

$= \ln 4 - \frac{1}{4} - (\ln 2 - \frac{1}{2})$

$= \ln 2 + \frac{1}{4}$

(d) $\underset{\sim}{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ & $\underset{\sim}{v} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

$\text{Proj}_{\underset{\sim}{u}} \underset{\sim}{v} = \dfrac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{\underset{\sim}{u} \cdot \underset{\sim}{u}} \underset{\sim}{u}$

$= \dfrac{2 \times -2 + 3 \times 4}{2^2 + 3^2} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$= \dfrac{8}{13}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ or $\dfrac{8}{13}(2\underset{\sim}{i} + 3\underset{\sim}{j})$

(e) $p = \frac{20}{100} = 0.2$, $q = 0.8$, $n = 100$

(i) Mean: $E(\hat{p}) = p = 0.2$

std dev: $Var(\hat{p}) = \frac{pq}{n}$

$$\sigma^2 = 0.0016$$

$$\sigma = 0.04$$

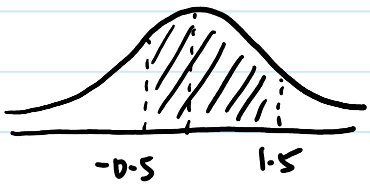
(ii) For $X = 18$, $z = \frac{0.18 - 0.2}{0.04} = -0.5$

$P(z > -0.5) = 1 - P(z < 0.5)$

$= 1 - 69.15$

$= 30.85$

for $X = 26$, $z = \frac{0.26 - 0.2}{0.04} = 1.5$



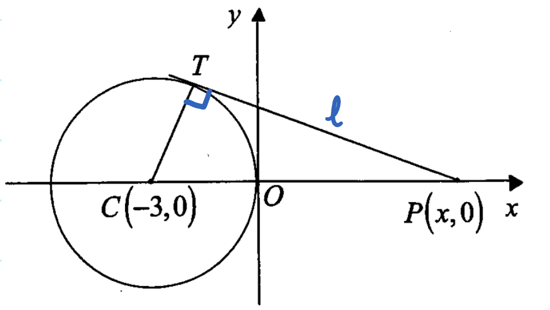
$P(z < 1.5) = 0.9332$

Then, $P(-0.5 < z < 1.5) = 0.9332 - 0.3085$

$= 0.6247$

Question 12

(a)



(i) $PC^2 = PT^2 + CT^2$

$(x+3)^2 = PT^2 + 3^2$

$PT^2 = x^2 + 6x + \cancel{9} - \cancel{9}$

$PT = \pm\sqrt{x^2+6x}$

But $PT > 0$, $\therefore PT = \sqrt{x^2+6x}$

(ii) Let $l = PT = (x^2+6x)^{1/2}$

$\frac{dl}{dt} = \frac{dl}{dx} \times \frac{dx}{dt}$

$\frac{dx}{dt} = 0.1$

$= \frac{1}{2}(x^2+6x)^{-1/2}(2x+6) \times 0.1$

$= \frac{x+3}{10\sqrt{x^2+6x}}$

Sub $x=2$,

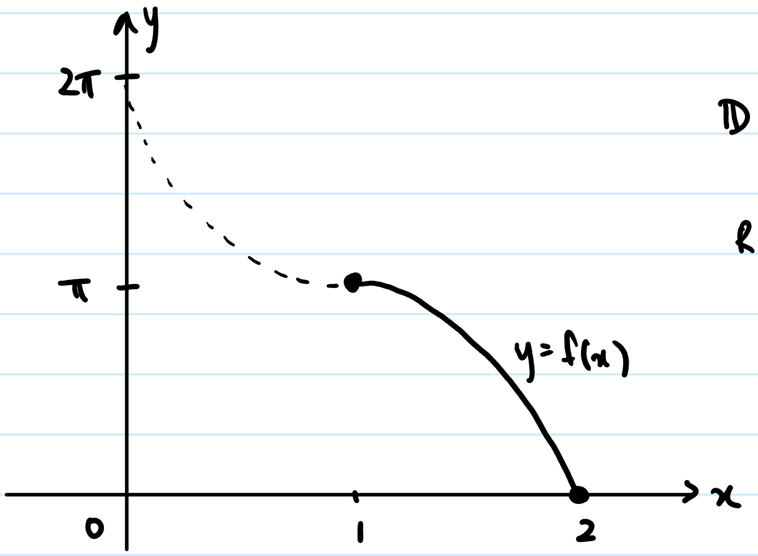
$\frac{dl}{dt} = \frac{2+3}{10\sqrt{2^2+6(2)}}$

$= 0.125$ cm/s

$\therefore$ PT increasing at 0.125 cm/s when $x=2$

(b) $f(x) = 2\cos^{-1}(x-1)$, $1 \le x \le 2$

(i)



D: $-1 \le x-1 \le 1$

$0 \le x \le 2$

R: $0 \le \cos^{-1}(x-1) \le \pi$

$0 \le 2\cos^{-1}(x-1) \le 2\pi$

$0 \le y \le 2\pi$

(ii) $f(x)$: $y = 2\cos^{-1}(x-1)$, $1 \le x \le 2$, $0 \le y \le \pi$

$f^{-1}(x)$: $x = 2\cos^{-1}(y-1)$

$\frac{x}{2} = \cos^{-1}(y-1)$

$\cos\left(\frac{x}{2}\right) = y-1$

$y = \cos\left(\frac{x}{2}\right)+1$

$\therefore f^{-1}(x) = \cos\left(\frac{x}{2}\right)+1$

D: $0 \le x \le \pi$

(c) $\underset{\sim}{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ & $\underset{\sim}{b} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

(i) $\underset{\sim}{a} + \underset{\sim}{b} = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$

$\underset{\sim}{a} - \underset{\sim}{b} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$

(ii) $(\underset{\sim}{a}+\underset{\sim}{b}) \cdot (\underset{\sim}{a}-\underset{\sim}{b}) = 7 \times -1 + 9 \times -5$

$= -52$

(iii) $(\underset{\sim}{a}+\underset{\sim}{b}) \cdot (\underset{\sim}{a}-\underset{\sim}{b}) \ne 0$

$\therefore$ Not perpendicular

(d) (i) $\int_0^1 \frac{\sqrt{x}}{(1+x)^2}\,dx$

Let $x = \tan^2\theta$

$dx = 2\tan\theta\sec^2\theta\,d\theta$

$\begin{cases} x=0, & \theta = 0 \\ x=1, & \theta = \pi/4 \end{cases}$

$= \int_0^{\pi/4} \frac{\tan\theta}{(1+\tan^2\theta)^2}\, 2\tan\theta\sec^2\theta\,d\theta$

$= \int_0^{\pi/4} \frac{2\tan^2\theta\,\cancel{\sec^2\theta}}{\cancel{\sec^4\theta}\ \sec^2\theta}\,d\theta$

$\sec^2\theta - \tan^2\theta = 1$

$\sec^2\theta = 1 + \tan^2\theta$

$= \int_0^{\pi/4} \frac{2\sin^2\theta / \cancel{\cos^2\theta}}{1/\cancel{\cos^2\theta}}\,d\theta$

$= \int_0^{\pi/4} 2\sin^2\theta\,d\theta$

(ii) $\int_0^1 \frac{\sqrt{x}}{(1+x)^2}\,dx = \int_0^{\pi/4} 2\sin^2\theta\,d\theta$

$= 2\int_0^{\pi/4} \frac{1}{2}(1-\cos 2\theta)\,d\theta$

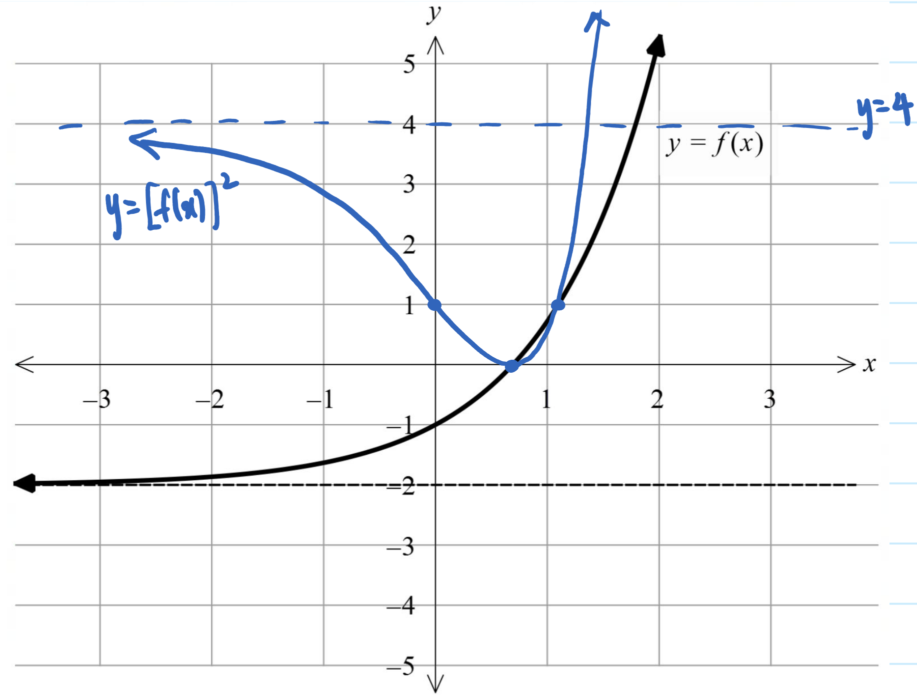
$= \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/4}$

$= \frac{\pi}{4} - \frac{\sin \pi/2}{2} - \left(0 - \frac{\sin 0}{2}\right)$

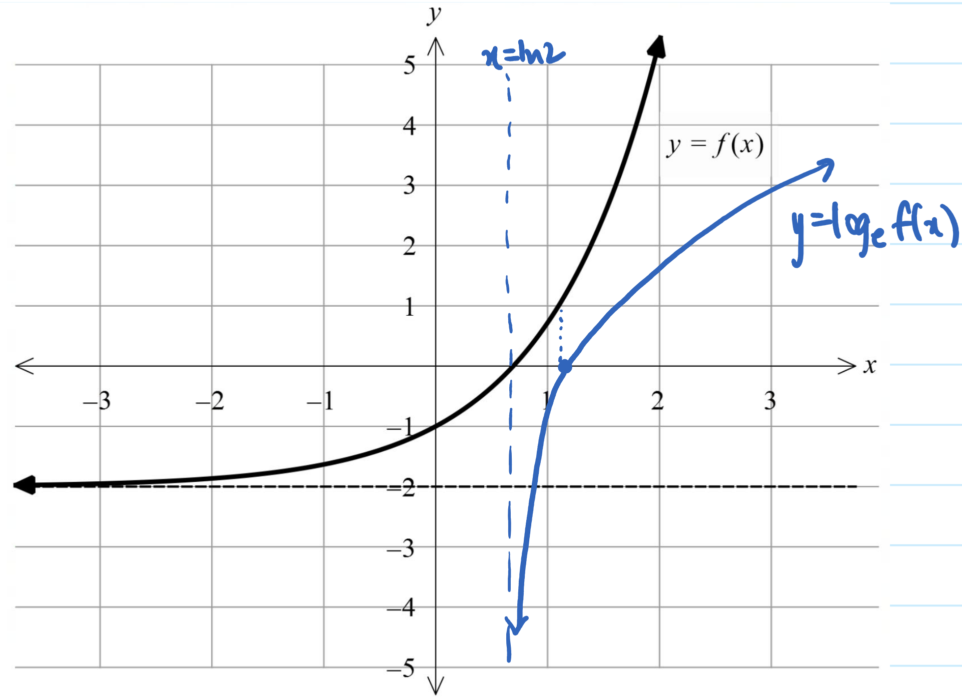
$= \frac{\pi}{4} - \frac{1}{2}$

Question 13

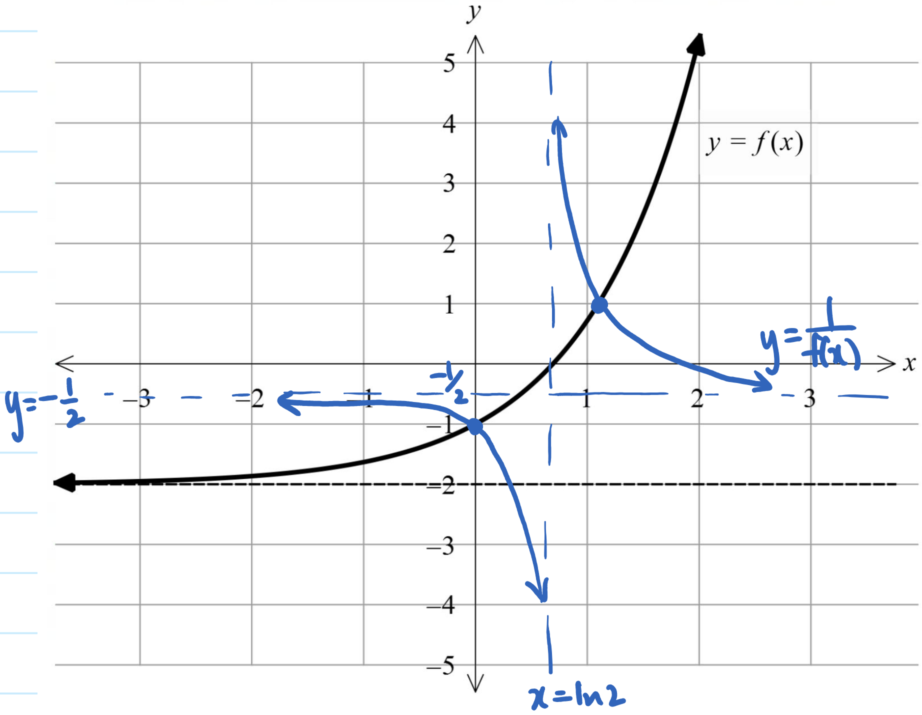
(a) (i) $y=[f(x)]^2$



(ii) $y=\log_e f(x)$

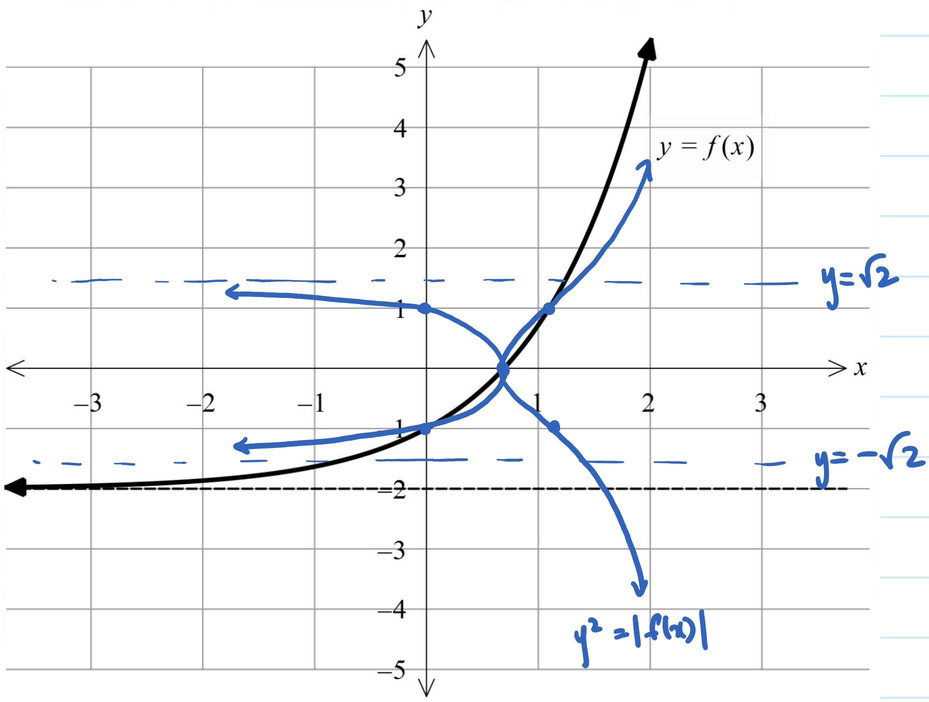


(iii) $y = \frac{1}{f(x)}$



(iv) $y^2 = |f(x)|$

$y = \pm\sqrt{|f(x)|}$



(b)(i) LHS $= 2\cos\left(x+\frac{\pi}{3}\right)$

$= 2\left(\cos x \cos\frac{\pi}{3} - \sin x \sin\frac{\pi}{3}\right)$

$= 2\left(\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x\right)$

$= \cos x - \sqrt{3}\sin x$

$=$ RHS

(ii) $\cos x - \sqrt{3}\sin x = \sqrt{3}$

$0 \le x \le 2\pi$

$\downarrow$

$\frac{\pi}{3} \le x+\frac{\pi}{3} \le \frac{7\pi}{3}$

$2\cos\left(x+\frac{\pi}{3}\right) = \sqrt{3}$

$\cos\left(x+\frac{\pi}{3}\right) = \sqrt{3}/2$

$\cos^{-1}\left(\sqrt{3}/2\right) = \pi/6$

$x+\frac{\pi}{3} = \cancel{\frac{\pi}{6}}, \frac{11\pi}{6}, \frac{13\pi}{6}$

$+2\pi$

$\therefore x = \frac{3\pi}{2}, \frac{11\pi}{6}$

(c)(i) Sub $T=-40+64e^{kt}$ in $\frac{dT}{dt} = k(T-T_0)$

LHS $= \frac{dT}{dt}$

$= \frac{d}{dt}\left(-40+64e^{kt}\right)$

$= 0 + 64ke^{kt}$

$= k(T+40)$

$= k(T--40)$

$= k(T-T_0)$, $T_0 = -40$

$=$ RHS

$T=-40+64e^{kt}$

$T+40 = 64e^{kt}$

(ii) Sub $t=5$, $T=19$

$19 = -40+64e^{5k}$

$e^{5k} = 59/64$

$5k = \ln\left(\frac{59}{64}\right)$

$k = \frac{1}{5}\ln\left(\frac{59}{64}\right)$

PTO $\to$

Sub $T=0$,

$$0=-40+64e^{kt}$$

$$40=64e^{kt}$$

$$e^{kt}=5/8$$

$$kt=\ln(5/8)$$

$$t=\frac{\ln(5/8)}{k}$$

$$=\frac{5\ln(5/8)}{\ln(59/64)}$$

$$=28.889\ldots$$

$$=29 \text{ seconds (nrst sec)}$$

(d) $f(x)=\cos^{-1}(2x-1)-2\cos^{-1}\sqrt{x}$

$$f'(x)=\frac{-(2x-1)'}{\sqrt{1-(2x-1)^2}}-\frac{-2(\sqrt{x})'}{\sqrt{1-(\sqrt{x})^2}}$$

$$=\frac{-2}{\sqrt{1-4x^2+4x-1}}+\frac{2\times\frac{1}{2\sqrt{x}}}{\sqrt{1-x}}$$

$$=\frac{-2}{\sqrt{4x(1-x)}}+\frac{1}{\sqrt{x}\times\sqrt{1-x}}$$

$$=\frac{-\cancel{2}}{\cancel{2}\sqrt{x(1-x)}}+\frac{1}{\sqrt{x(1-x)}}$$

$$=\frac{-1}{\sqrt{x(1-x)}}+\frac{1}{\sqrt{x(1-x)}}$$

$$=0$$

Question 14

(a) RTP: $2\times1+3\times2+4\times3+\ldots+n(n-1)=\frac{1}{3}n(n^2-1)$, $n\geq2$

Step 1: For $n=2$

$LHS=2\times1=2$

$RHS=\frac{1}{3}\times2\times(2^2-1)=2$

$\therefore LHS=RHS$, statement is true for $n=2$.

Step 2: Assume statement is true for $n=k$, $k\geq2$ integer

i.e. $2\times1+3\times2+4\times3+\ldots+k(k-1)=\frac{1}{3}k(k^2-1)$ (*)

Step 3: Attempt to prove statement is true for $n=k+1$

i.e. $2\times1+3\times2+4\times3+\ldots+(k+1)k=\frac{1}{3}(k+1)\left[(k+1)^2-1\right]$

$=\frac{1}{3}(k+1)(k^2+2k)$

$LHS=\underline{2\times1+3\times2+4\times3+\ldots+k(k-1)}+(k+1)k$

$=\underline{\frac{1}{3}k(k^2-1)}+(k+1)k$ from assumption (*)

$=\frac{1}{3}k(k+1)(k-1)+k(k+1)$

$=\frac{1}{3}(k+1)\left[k(k-1)+3k\right]$

$=\frac{1}{3}(k+1)(k^2-k+3k)$

$=\frac{1}{3}(k+1)(k^2+2k)$

$=RHS$

$\therefore$ Statement is true for $n=k+1$ if it is true for $n=k$.

Step 4 By the principle of mathematical induction, the statement is true for integers $n\geq2$.

(b) (i) Sub $x=\frac{\pi}{4}$

① $y=\sec\frac{\pi}{4}$

$=\sqrt{2}$

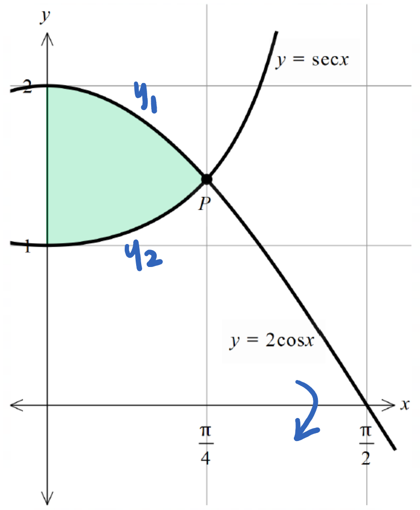
② $y=2\cos\frac{\pi}{4}$

$=2\times\frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$

$=\sqrt{2}$

$\therefore$ POI at $\left(\frac{\pi}{4},\sqrt{2}\right)$

(ii) $V=\pi\int_a^b y_1^2-y_2^2\,dx$



$=\pi\int_0^{\pi/4}(2\cos x)^2-(\sec x)^2\,dx$

$=\pi\int_0^{\pi/4}4\cos^2 x-\sec^2 x\,dx$

$=\pi\int_0^{\pi/4}4\times\frac{1}{2}(1+\cos 2x)-\sec^2 x\,dx$

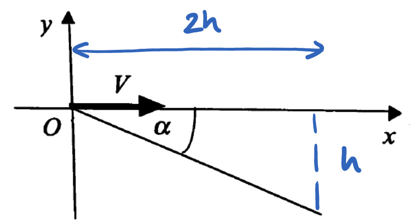
$=\pi\left[2\left(x+\frac{\sin 2x}{2}\right)-\tan x\right]_0^{\pi/4}$

$=\pi\left[\left(\frac{\pi}{2}+\sin\frac{\pi}{2}-\tan\frac{\pi}{4}\right)-\left(0+\frac{\sin 0}{2}-\tan 0\right)\right]$

$=\pi\left(\frac{\pi}{2}+1-1\right)$

$=\frac{\pi^2}{2}$ units^3

(c)



$\alpha = \tan^{-1}\left(\frac{1}{2}\right)$

$\tan\alpha = \frac{1}{2} = m$

$$\begin{cases} x = vt & \text{— ①} \\ y = -\frac{1}{2}gt^2 & \text{— ②} \end{cases}$$

$y = mx$

$y = \frac{1}{2}x$ — ③

(i) Sub ① & ② in ③

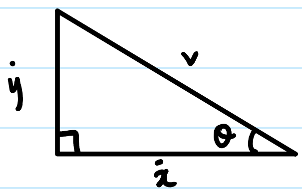
$-\frac{1}{2}gt^2 = \frac{1}{2}vt$

$gt = v$

$\therefore t = \frac{v}{g}$ seconds

(ii) $\dot{x} = V$

$\dot{y} = -gt = V$ when $t = V/g$



$v = \sqrt{\dot{x}^2 + \dot{y}^2}$

$= \sqrt{V^2 + V^2}$

$= \sqrt{2V^2}$

$= \sqrt{2}\,V$ m/s

$\tan\theta = \dot{y}/\dot{x}$

$= V/V$

$= 1$

$\theta = \tan^{-1}(1) = 45^\circ$

(d)(i) $(1-x)^{2n} = \binom{2n}{0} - \binom{2n}{1}x + \binom{2n}{2}x^2 - \binom{2n}{3}x^3 + \dots + (-1)^n\binom{2n}{n}x^n + \dots + \binom{2n}{2n}x^{2n}$

(ii) Differentiate both sides WRT x

$2n(1-x)^{2n-1} = -\binom{2n}{1} + 2\binom{2n}{2}x - 3\binom{2n}{3}x^2 + \dots + n(-1)^{n-1}\binom{2n}{n}x^{n-1} + \dots + 2n\binom{2n}{n}x^{2n-1}$

Let $x = 1$

$0 = -\binom{2n}{1} + 2\binom{2n}{2} - 3\binom{2n}{3} + \dots + n(-1)^{n-1}\binom{2n}{n} + \dots + 2n\binom{2n}{2n}$

Add "odd" terms to both sides

$\therefore \binom{2n}{1} + 3\binom{2n}{3} + 5\binom{2n}{5} + \dots + (2n-1)\binom{2n}{2n-1} = 2\binom{2n}{2} + 4\binom{2n}{4} + \dots + 2n\binom{2n}{2n}$